

## DIRECT LEAST-SQUARES ELLIPSE FITTING

J. Courtney and A. M. de Paor  
 Dept. of Electronic and Electrical Engineering,  
 University College Dublin,  
 Belfield, Dublin 4, Rep. of Ireland.  
 E-mail: Jane.Courtney@ucd.ie

### Abstract

Many biological and astronomical forms can be best represented by ellipses. While some more complex curves might represent the shape more accurately, ellipses have the advantage that they are easily parameterised and define the location, orientation and dimensions of the data more clearly. In this paper, we present a method of direct least-squares ellipse fitting by solving a generalised eigensystem. This is more efficient and more accurate than many alternative approaches to the ellipse-fitting problem such as fuzzy c-shells clustering and Hough transforms. This method was developed for human body modelling as part of a larger project to design a marker-free gait analysis system which is being undertaken at the National Rehabilitation Hospital, Dublin.

**Key Words:** modelling in Biomedicine and Biomechanics; data modelling; limb modelling

### 1. Introduction

#### 1.1. Problem Statement

A general conic can be represented by the parameters of its characteristic polynomial. Consider the general second-order polynomial:

$$f(\mathbf{a}, \mathbf{x}) = a_0x^2 + a_1xy + a_2y^2 + a_3x + a_4y + a_5 = \mathbf{a} \cdot \mathbf{x} \quad \dots(1)$$

where  $\mathbf{a} = (a_0 \ a_1 \ a_2 \ a_3 \ a_4 \ a_5)^T$

and  $\mathbf{x} = (x^2 \ xy \ y^2 \ x \ y \ 1)^T$

$f(\mathbf{a}, \mathbf{x}) = 0$  is the equation of the conic and  $f(\mathbf{a}, \mathbf{x}_i)$  is a measure of the distance from point  $\mathbf{p}_i = (x_i, y_i)$  to the curve. Given a data set of  $N$  points, least-squares fitting involves minimising the sum of the squared distance of the curve to each of the  $N$  points. By using  $f(\mathbf{a}, \mathbf{x})$  as our distance measure, this problem can be stated as the

minimisation with respect to  $\mathbf{a}$  of the following objective function

$$d(\mathbf{a}) = \sum_{i=1}^N f^2(\mathbf{a}, \mathbf{x}_i) \quad (2)$$

We can re-state this problem in matrix form and apply a constraint which will limit the solution to an elliptic curve. Thus we can solve for the vector of parameters,  $\mathbf{a}$ .

#### 1.2. Literature Review

In a seminal paper, Bookstein [1] introduced an idea for fitting shapes to scattered data. He reduced the problem of the above minimisation to solving a rank-deficient generalised eigensystem for the parameters of the characteristic equation with respect to a quadratic constraint. Many constraints were suggested in previous papers [2] [3] and [4], both for general conics and for ellipses. However, in Fitzgibbon's ellipse-specific implementation of Bookstein's method [5], the equality constraint (3) is found to be acceptable.

$$4a_0a_2 - a_1^2 = 1 \quad (3)$$

In Fitzgibbon's work, the block decomposition used to solve for the parameters was abandoned and a general eigensystem solver was used. In seeking a more efficient solution (our method will later be used in motion tracking where time is of the essence!), we have made use of some of the properties of this ellipse-specific problem and have developed an efficient algorithm for direct fitting of ellipses.

### 2. Gathering Data

In a typical ellipse fitting problem, the data will be in the form of points on a shape outline or a cluster of points in the  $xy$ -plane. We firstly gather our data in the form of a *design matrix*  $\mathbf{D}$  where  $\mathbf{D} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_N]^T$  and the  $\mathbf{x}_i$  are as defined previously. So  $\mathbf{D}$  is of the form

$$\mathbf{D} = \begin{pmatrix} x_1^2 & y_1^2 & x_1 y_1 & x_1 & y_1 & 1 \\ x_2^2 & y_2^2 & x_2 y_2 & x_2 & y_2 & 1 \\ x_3^2 & y_3^2 & x_3 y_3 & x_3 & y_3 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_N^2 & y_N^2 & x_N y_N & x_N & y_N & 1 \end{pmatrix} \quad (4)$$

where  $x_i$  and  $y_i$  are the x and y co-ordinates of the data points. The objective function  $d(\mathbf{a})$  can be re-written in matrix form as

$$d(\mathbf{a}) = \mathbf{a}^T \mathbf{D}^T \mathbf{D} \mathbf{a} = \mathbf{a}^T \mathbf{S} \mathbf{a} \quad (5)$$

where  $\mathbf{S}$  is the 6x6 *scatter matrix*,  $\mathbf{S} = \mathbf{D}^T \mathbf{D}$ . Expression (3) can be written as  $\mathbf{a}^T \mathbf{C} \mathbf{a} = 1$  where  $\mathbf{C}$  is the *constraint matrix*

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now our minimisation problem can be re-defined in matrix form by introducing the Lagrange multiplier and differentiating. Our new eigensystem is now

$$\begin{aligned} \mathbf{S} \mathbf{a} &= \lambda \mathbf{C} \mathbf{a} & (6) \\ \mathbf{a}^T \mathbf{C} \mathbf{a} &= 1 & (7) \end{aligned}$$

$\mathbf{S}$  is positive definite and there is a unique solution which gives the best fit ellipse.

### 3. Block Decomposition

Because of the structure of the constraint matrix,  $\mathbf{C}$ , in this unique problem, the complexity can be reduced and the problem simplified by breaking the matrices up into 3x3 blocks:

$$\begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{C}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} \quad (8)$$

where  $\mathbf{a}_1 = (a_0 \ a_1 \ a_2)^T$  and  $\mathbf{a}_2 = (a_3 \ a_4 \ a_5)^T$ . This gives a system of equations

$$\begin{aligned} \mathbf{S}_{11} \mathbf{a}_1 + \mathbf{S}_{12} \mathbf{a}_2 &= \lambda \mathbf{C}_{11} \mathbf{a}_1 & (9) \\ \mathbf{S}_{21} \mathbf{a}_1 + \mathbf{S}_{22} \mathbf{a}_2 &= \mathbf{0} & (10) \end{aligned}$$

Solving for  $\mathbf{a}_2$  in terms of  $\mathbf{a}_1$  gives

$$\mathbf{a}_2 = -\mathbf{S}_{22}^{-1} \mathbf{S}_{12}^T \mathbf{a}_1 \quad (11)$$

Given  $\mathbf{S}$  positive definite, we can deduce that  $\mathbf{S}_{22}$  is non-singular and also positive definite. Because  $\mathbf{S}$  is symmetric,  $\mathbf{S}_{21} = \mathbf{S}_{12}^T$ . Now, the problem of solving for  $\mathbf{a}_1$  reduces down to the equation

$$[\lambda \mathbf{I} - \mathbf{E}] \mathbf{a}_1 = \mathbf{0} \quad (12)$$

where  $\mathbf{I}$  is the 3x3 identity matrix and  $\mathbf{E}$  is given by

$$\mathbf{E} = \mathbf{C}_{11}^{-1} [\mathbf{S}_{11} - \mathbf{S}_{12} \mathbf{S}_{22}^{-1} \mathbf{S}_{12}^T] \quad (13)$$

This shows that  $\lambda$  is an eigenvalue of  $\mathbf{E}$ . If we look at the original equation and pre-multiply it by  $\mathbf{a}^T$  we get

$$\mathbf{a}^T \mathbf{S} \mathbf{a} = \lambda \mathbf{a}^T \mathbf{C} \mathbf{a} \quad (14)$$

The left-hand side of this equation is positive since  $\mathbf{S}$  is positive definite and the right-hand side is equal to  $\lambda$  since  $\mathbf{a}^T \mathbf{C} \mathbf{a} = 1$ . Therefore, we must have  $\lambda > 0$ .

It can be shown that  $\mathbf{E}$  has only one positive, real eigenvalue and the other two are real and negative (see Appendix I). It is clear from (12) that  $\mathbf{a}_1$  is the eigenvector corresponding to this unique, positive eigenvalue.

To find the one positive eigenvalue, we derive the characteristic polynomial of  $\mathbf{E}$  and solve for its roots. As this is a cubic polynomial and we are only interested in its sole positive real root, we can easily solve for our desired eigenvalue (see Appendix II). Now, we determine the corresponding eigenvector,  $\mathbf{v}$ . Note that if  $\mathbf{v}$  is an eigenvector, then  $k\mathbf{v}$  is also an eigenvector if  $k$  is a constant. Therefore, to solve for  $\mathbf{a}_1$ , we must determine the appropriate scaling factor  $k$ .

The constraint equation (7) can also be broken down by block decomposition to the equation

$$\mathbf{a}_1^T \mathbf{C}_{11} \mathbf{a}_1 = 1 \quad (15)$$

So, in order to meet the constraint, we must scale our general eigenvector,  $\mathbf{v}$  to give the specific eigenvector  $\mathbf{a}_1$  for which this equation holds. If we write  $\mathbf{a}_1 = k\mathbf{v}$  and substitute this into (15), we find an equation for  $k$ :

$$k = \sqrt{\frac{1}{\mathbf{v}^T \mathbf{C}_{11} \mathbf{v}}} \quad (16)$$

We now have a solution for  $\mathbf{a}_1$  and we can solve for  $\mathbf{a}_2$  from (11), so we can solve for  $\mathbf{a}$ , giving us a complete, unique, least-squares solution to our ellipse-fitting problem.

#### 4. Parameterizing the Ellipse

Once the equation for the ellipse has been found by solving for the vector of coefficients,  $\mathbf{a}$ , we need to determine the parameters of the ellipse, i.e. the position, orientation and dimensions. We find the position (the centre of the ellipse) by simply minimising the equation of the ellipse with respect to  $x$  and  $y$ . This gives a pair of equations which can be solved simultaneously to give  $x_c$  and  $y_c$ , the co-ordinates of the centre:

$$(x_c, y_c) = (x, y) @ \frac{df(x,y)}{dx} = 0 \text{ and } \frac{df(x,y)}{dy} = 0 \quad \dots(17)$$

In order to determine the major and minor axes, we translate the ellipse to centre it at the origin. This eliminates the  $x$  and  $y$  terms in the equation so the equation of the ellipse can be rewritten in matrix form:

$$\mathbf{x}^T \mathbf{M} \mathbf{x} = c \quad (18)$$

where  $\mathbf{x}^T = (x, y)$ ,

$$\mathbf{M} = \begin{pmatrix} a_0 & \frac{a_1}{2} \\ \frac{a_1}{2} & a_2 \end{pmatrix}$$

and  $c$  is a constant,  $c = -f(x_c, y_c)$ .

By using Lagrange multipliers, we find that the problem of finding the maximum and minimum points on the ellipse reduces to solving for the eigenvalues and eigenvectors of  $\mathbf{M}$ . If we consider  $\mathbf{v}$ , an eigenvector of  $\mathbf{M}$ , we can say that our solution  $\mathbf{x}$  is also an eigenvector so  $\mathbf{x} = k\mathbf{v}$ , where  $k$  is some constant. Now we solve for  $k$ , as before, by substituting this relationship into the ellipse equation, (18):

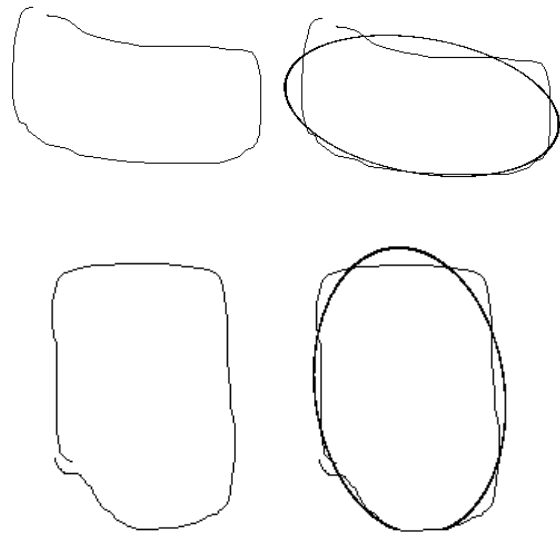
$$k = \sqrt{\frac{c}{\mathbf{v}^T \mathbf{M} \mathbf{v}}} \quad (19)$$

Now the two solutions for  $\mathbf{x}$  (one for each eigenvalue) are the vectors corresponding to the major and minor axes of the ellipse. Finally, finding the orientation is a trivial problem. We simply find the angle that the major axis makes with the horizontal.

#### 5. Results

Comparative results have already been given for least-squares ellipse-fitting in [5] and so need not be repeated here. This method has already been shown to be robust, accurate and reliable when compared to other least-squares based methods (see [6] and [7]). Least-squares

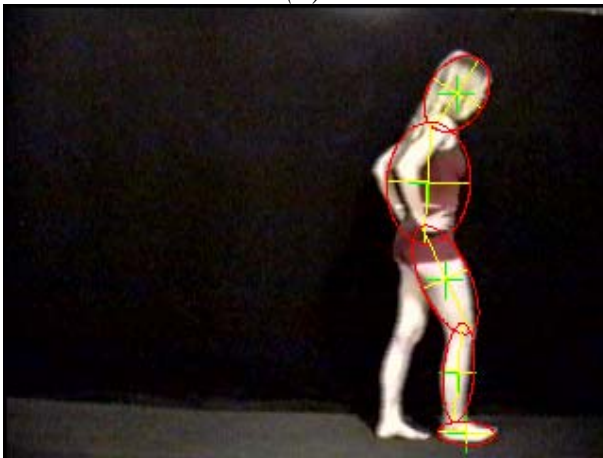
methods are found to be preferable in many cases to other available methods such as Hough transforms [8], moments [9] or fuzzy c-shells clustering [10] due to their low computational costs and overall efficiency. By using the block decomposition and availing of the unique characteristics of the ellipse-specific problem, we add to the efficiency of the algorithm. Our particular algorithm is efficient enough to be used, not only in multiple ellipse fitting cases (e.g. human body modelling) but also in video-based motion tracking.



**FIGURE 1: Ellipse-fitting performed on crudely-drawn ellipses**



(a)



(b)

**FIGURE 2: Ellipse-fitting in action in human body modelling (a) original image; (b) image with ellipses fit to body segments (note: the torso is of minimal interest in gait analysis and so is modelled by a single ellipse)**

## References

- [1] F.L. Bookstein, "Fitting Conic Sections to Scattered Data," *Computer Graphics and Image Processing*, no. 9, pp. 56-71, 1979.
- [2] T. Ellis, A. Abbood, and B. Brillault, "Ellipse Detection and Matching With Uncertainty," *Image and Vision Computing*, vol. 10, no. 2, pp. 271-276, 1992.
- [3] P.L. Rosin, "A Note on the Least Squares Fitting of Ellipses," *Pattern Recognition Letters*, no. 14, pp. 799-808, Oct. 1993.
- [4] J. Porrill, "Fitting Ellipses and Predicting Confidence Envelopes Using a Bias Corrected Kalman Filter," *Image and Vision Computing*, vol. 8, no. 1, pp. 37-41, Feb. 1990.

[5] A. Fitzgibbon, M. Pilu, R. Fisher, "Direct Least Square Fitting of Ellipses," *IEEE Transactions On Pattern Analysis And Machine Intelligence*, Vol. 21, No. 5, May 1999

[6] W. Gander, G.H. Golub, and R. Strebler, "Least-Square Fitting of Circles and Ellipses," *BIT*, no. 43, pp. 558-578, 1994.

[7] G. Taubin, "Estimation of Planar Curves, Surfaces and Non-Planar Space Curves Defined by Implicit Equations, With Applications to Edge and Range Image Segmentation," *IEEE Trans. Pattern Analysis and Machine Intelligence*, vol. 13, no. 11, pp. 1,115-1,138, Nov. 1991.

[8] V.F. Leavers, *Shape Detection in Computer Vision Using the Hough Transform*. New York: Springer-Verlag, 1992.

[9] S. Paschalakis, P. Lee, "Pattern recognition in grey level images using moment based invariant features," *Seventh International Conference on Image Processing And Its Applications*, vol.1, pp. 245-249, Jul 1999.

[10] R.N. Dave, K. Bhaswan, "Adaptive Fuzzy C-Shells Clustering and Detection of Ellipses," *IEEE Trans. on Neural Networks*, vol. 3, no. 5, Sept 1992.